Shape derivative of the Cheeger constant

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SHAPE DERIVATIVE OF THE CHEEGER CONSTANT

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ABSTRACT. This paper deals with the existence of the shape derivative of the Cheeger constant $h_1(\Omega)$ of a bounded domain Ω. We prove that if Ω admits a unique Cheeger set, then the shape derivative of $h_1(\Omega)$ exists, and we provide an explicit formula. A counter-example shows that the shape derivative may not exist without the uniqueness assumption.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The *Cheeger constant* of Ω is defined as

$$h_1(\Omega) := \inf_{E \subset \Omega} \frac{P(E;\mathbb{R}^n)}{|E|}.$$

Here $P(E;\mathbb{R}^n)$ is the distributional perimeter of E measured with respect to $\mathbb{R}^n$, while $|E|$ is the $n-$dimensional Lebesgue measure of E. A set $C \subset \Omega$ for which the infimum is attained is called a *Cheeger set*.

The problem of finding a Cheeger set for a given domain Ω has extensively received attention in the last decades, starting from the original work of Jeff Cheeger [5]. For an introductory survey on the Cheeger problem we refer to [18]; here we recall that for every bounded domain Ω with Lipschitz boundary there exists at least one Cheeger set. Uniqueness does not hold in general, but it is guaranteed if we assume Ω to be convex; in this case the Cheeger set turns out to be convex and of class $C^{1,1}$ (see [1]). The Cheeger constant can be obtained as the limit for $p \to 1$ of the first eigenvalue $\lambda_p(\Omega)$ of the $p-$Laplacian under Dirichlet boundary conditions (see [12]), and corresponds to the first eigenvalue of the $1-$Laplacian (see [14]).

Shape analysis roughly consists in studying the regularity and the optimisation of a functional $J : \Omega \in \mathscr{A} \to J(\Omega) \in \mathbb{R}$ defined over some class $\mathscr{A}$ of subsets $\Omega \subset \mathbb{R}^n$. Due to its physical relevance, a particularly important class of functionals are the ones defined in terms of the eigenvalues of some operator. A lot of works have been dedicated for instance to the study of the dependence of the eigenvalues of the Laplacian as functions of the domain under various boundary conditions. We refer for example to the monograph [11] for an introduction to the field of shape analysis.

In order to optimize J over $\mathscr{A}$ it is important to determine how sensitive is J under perturbation of a given set Ω. Given a smooth vector field $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, define $F_t : \mathbb{R}^n \to \mathbb{R}^n$ as $F_t(x) = (Id + tV)(x)$. We then perturb Ω in the direction V by considering the sets $\Omega_t = F_t(\Omega)$. The shape derivative of J in the direction V at Ω is then defined as

$$J(\Omega,V)' := \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

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For instance the shape derivative of the first eigenvalue $\lambda(\Omega)$ of the Laplacian with Dirichlet boundary condition is

$$\lambda(\Omega,V)' = -\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \langle V,\nu\rangle \, d\mathscr{H}^{n-1},$$

where u is the unique positive normalized eigenfunction in Ω and ν is the unit exterior normal to $\partial\Omega$. This formula has been generalized in [8, 16] to the first eigenvalue $\lambda_p(\Omega)$ of the p-Laplacian ($p>1$):

$$\lambda_p(\Omega,V)' = -(p-1)\int_{\partial\Omega} \left|\frac{\partial u_p}{\partial \nu}\right|^p \langle V,\nu\rangle \, d\mathscr{H}^{n-1}, \tag{1}$$

where u_p is the unique positive normalized eigenfunction in Ω.

General results about the stability of the Cheeger constant $h_1(\Omega)$ as a function of Ω have been obtained in [10]. In particular the shape derivative was computed but only in the case $V(x)=\lambda x$, $\lambda\in\mathbb{R}$. The main purpose of this paper is to provide a formula for the shape derivative of $h_1(\Omega)$ in the case of an arbitrary deformation field V. Notice that setting $p=1$ formally in (1) does not give any meaningful information. Indeed it is known that characteristic functions of Cheeger sets are, up to a multiplicative constant, normalized first eigenfunctions of the 1-Laplacian and they are obtained as limit of eigenfunctions of the p-Laplacian as p goes to 1 (see Section 2). Therefore, if C is a Cheeger set, the normal derivative should be thought as equal to $-\infty$ on $\partial\Omega\cap\partial C$, so that the integral in (1) would be infinite. This kind of problem has also been considered in [20] where the shape derivative of the best Sobolev constant for the embedding of $BV(\Omega)$ into $L^1(\partial\Omega)$ was computed. Let us mention finally that the other extreme case $p=+\infty$ corresponding to the first eigenvalue of the ∞-Laplacian has been recently studied in [17], [7] and [19] for Dirichlet, Steklov and Neumann boundary condition respectively.

The main result of our paper is the following.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain. Let $V\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, and let $F_t:\mathbb{R}^n\to\mathbb{R}^n$ be the one-parameter family of diffeomorphisms defined by $F_t(x)=(Id+tV)(x)$. Set $\Omega_t=F_t(\Omega)$. Then*

$$\lim_{t\to 0} h_1(\Omega_t) = h_1(\Omega).$$

If moreover Ω admits a unique Cheeger set C then the shape derivative

$$h_1(\Omega,V)' = \lim_{t\to 0}\frac{h_1(\Omega_t)-h_1(\Omega)}{t}$$

exists and is given by

$$h_1(\Omega,V)' = \frac{1}{|C|}\int_{\partial^* C} \left(div_{\partial C} V - h_1(\Omega)\langle V,\nu\rangle\right) d\mathscr{H}^{n-1}, \tag{2}$$

where $\partial^* C$ is the reduced boundary of C, ν is the unit exterior normal vector on $\partial^* C$, and $div_{\partial\Omega} V(x) = div V(x) - (\nu(x), DV(x)\nu(x))$, $x\in\partial^*\Omega$, is the tangential divergence of V on $\partial\Omega$.

In the case where ∂C is of class $C^{1,1}$, this formula can be simplified:

Corollary 1.2. *If Ω admits a unique Cheeger set C and ∂C is of class $C^{1,1}$, then the shape derivative of $h_1(\Omega)$ is given by the formula*

$$h_1(\Omega,V)' = \frac{1}{|C|}\int_{\partial C\cap\partial\Omega} (\kappa - h_1(\Omega))\langle V,\nu\rangle \, d\mathscr{H}^{n-1}, \tag{3}$$

where $\kappa(x) = div\ \nu$ is the sum of the principal curvatures of $\partial\Omega$ at the point x (i.e. $(n-1)$ times the mean curvature), and ν is the unit exterior normal to $\partial\Omega$.

The assumption in the Corollary is in particular satisfied for every dimension n when Ω is convex (see [1]), or in dimension $n \le 7$ when $\partial\Omega$ is of class $C^{1,1}$ and admits a unique Cheeger set C (see [4]). We point out that the uniqueness hypothesis is necessary. Indeed, at the end of this paper we provide a counterexample of a domain admitting more than one Cheeger set, which is not shape differentiable for some choice of V. However, it is interesting to observe that the bounded domains Ω admitting a unique Cheeger set (and hence shape differentiable) are dense in the L^1 topology (see [4]).

2. Definitions and preliminary results

Let $\Omega \subset \mathbb{R}^n$ be an open set. The *total variation* in Ω of a function $u \in L^1(\Omega)$ is defined as

$$|Du|(\Omega) := \sup\left\{ \int_\Omega u \,\mathrm{div}\varphi \,\middle|\, \varphi \in C^1_c(\Omega;\mathbb{R}^n),\, \|\varphi\|_\infty \le 1 \right\}.$$

A function u such that $|Du|(\Omega) < +\infty$ is said to be of *bounded variation*. The space of the functions of bounded variation will be denoted by $BV(\Omega)$. It can be easily proved that the total variation is lower semicontinuous with respect to the L^1-convergence (see [9]). Moreover, the following holds true. Suppose that Ω is a Lipschitz domain, and let $u \in BV(\Omega)$; if we denote by $\overline{u}$ the extension of u by zero outside Ω, then $\overline{u} \in BV(\mathbb{R}^n)$, and

$$|D\overline{u}|(\mathbb{R}^n) = |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathscr{H}^{n-1},$$

where $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$.

The *perimeter* of a set $E \subset \Omega$ (measured with respect to $\mathbb{R}^n$) is defined as

$$P(E;\mathbb{R}^n) := |D\chi_E|(\mathbb{R}^n),$$

where χ_E is the characteristic function of E. The *Cheeger constant* of Ω is

$$h_1(\Omega) := \inf_{E\subset\Omega} \frac{P(E;\mathbb{R}^n)}{|E|},$$

where $|E|$ stands for the n-dimensional Lebesgue measure of E. A *Cheeger set* is a set $C \subset \Omega$ such that

$$\frac{P(C;\mathbb{R}^n)}{|C|} = h_1(\Omega).$$

The existence of a Cheeger set for every bounded Lipschitz domain Ω is proved via the direct method of the Calculus of Variations. Uniqueness does not hold in general; however, any convex body has a unique Cheeger set (see [1]). If C is a Cheeger set for Ω, then $\partial C \cap \Omega$ is analytic, up to a closed singular set of Hausdorff dimension $n-8$; at the regular points of $\partial C \cap \Omega$, the mean curvature is equal to $\frac{h_1(\Omega)}{n-1}$ (see e.g. [18, Proposition 4.2]). Morever, if $\partial\Omega$ is of class $C^{1,1}$, then also ∂C enjoys the same regularity (see [4]); the same result holds if Ω is convex, as a consequence of the results in [21].

As an application of the coarea formula, $h_1(\Omega)$ can also be obtained as

$$h_1(\Omega) = \inf_{u\in BV(\Omega)\setminus\{0\}} \frac{|D\overline{u}|(\mathbb{R}^n)}{\|u\|_1}$$

or equivalently

$$h_1(\Omega) = \inf\{|D\overline{u}|(\mathbb{R}^n) \,|\, u \in BV(\Omega),\, \|u\|_1 = 1\}.$$

Therefore, $h_1(\Omega)$ can be seen as the first eigenvalue of the 1-Laplacian with Dirichlet boundary condition, which is defined formally as

$$\Delta_1 u = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right),$$

and the characteristic functions of Cheeger sets are corresponding eigenfunctions. We refer to [14] for a thorough analysis of this problem. Here we observe that if Ω admits a unique Cheeger set C, then $u = \frac{1}{|C|}\chi_C$ is the unique nonnegative normalized eigenfunction of the 1-Laplacian, since every level set of a first eigenfunction is a Cheeger set (see [3, Theorem 2]).

3. Proof of the main results

Recall that we are given a Lipschitz domain $\Omega \subset \mathbb{R}^n$ that we perturb in the direction of a smooth vector field $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ in the sense that we consider the perturbed domains

$$\Omega_t = F_t(\Omega) \qquad \text{with} \qquad F_t(x) = (Id + tV)(x).$$

We let $h = h_1(\Omega)$ and $h_t = h_1(\Omega_t)$. We also assume that any function u defined in Ω (resp. Ω_t) is extended by 0 to $\mathbb{R}^n\setminus\overline{\Omega}$ (resp. $\mathbb{R}^n\setminus\overline{\Omega_t}$). With the notation of the previous section this means that $u = \bar{u}$.

We recall the change of variable formula for BV functions (see [9, Lemma 10.1]). Let G_t be the inverse of F_t (which exists for small t). For an arbitrary function $u \in BV(\Omega)$, if we denote by v the function of $BV(\Omega_t)$ defined by $v(x) = u(G_t(x))$ we have the relations

$$\int_{\Omega_t} v(x)\,dx = \int_\Omega u(y)|\det DF_t(y)|\,dy$$

and

$$|Dv|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |(DG_t)^T\sigma|\cdot|\det DF_t|\,d|Du|,$$

where σ comes from the polar decomposition $Du = \sigma|Du|$.

Proof of Theorem 1.1. Let $u \in BV(\Omega)$ be a nonnegative eigenfunction for h such that $\|u\|_1 = 1$ in the sense that u is an extremal in (2) (which is known to exist). Consider the function $w_t \in BV(\Omega_t)$ defined as $w_t = u\circ G_t$. Then

$$|Dw_t|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |(DG_t)^T\sigma|\cdot|\det DF_t|\,d|Du|,$$

where σ comes from the polar decomposition $Du = \sigma|Du|$. Since $|\sigma| = 1$ $|\nabla u|$-a.e., and $DF_t \to Id$ uniformly as $t \to 0$, so that $|\det DF_t| \to 1$ uniformly, we have using (2) and the above change of variable formula that

$$h_t \le \frac{|Dw_t|(\mathbb{R}^n)}{\displaystyle\int_{\Omega_t} w_t} = \frac{\displaystyle\int_{\mathbb{R}^n} |(DG_t)^T|\cdot|\det DF_t|\,d|Du|}{\displaystyle\int_\Omega u(y)|\det DF_t(y)|\,dy} = (1+o(1))\frac{\displaystyle\int_{\mathbb{R}^n} d|Du|}{\displaystyle\int_\Omega u(y)\,dy}.$$

It follows that

$$\limsup_{t\to 0} h_t \le h$$

Let $u_t \in BV(\Omega_t)$ be a nonnegative extremal for h_t such that $\|u_t\|_1 = 1$. Consider the function $v_t \in BV(\Omega)$ defined as $v_t = u_t \circ F_t$. Then

$$
\begin{aligned}
|Dv_t|(\mathbb{R}^n) &= \int_{\mathbb{R}^n} |(DF_t)^T \sigma_t| \cdot |\det DG_t|\, d|Du_t| \leq (1+o(1)) \int_{\mathbb{R}^n} d|Du_t| \\
&= (1+o(1))h_t \\
&\leq h + o(1),
\end{aligned}
\tag{4}
$$

and

$$
\int_\Omega v_t\, dx = \int_{\Omega_t} u_t |\det DF_t^{-1}|\, dx = 1 + o(1). \tag{5}
$$

Therefore (v_t) is bounded in $BV(\mathbb{R}^n)$. Since the embedding of $BV(\mathbb{R}^n)$ into $L^1_{loc}(\mathbb{R}^n)$ is compact, it follows that there exists a function $v \in BV(\mathbb{R}^n)$ such that (up to a subsequence), $v_t \to v$ a.e.. We deduce first that $v = 0$ in $\mathbb{R}^n \backslash \Omega$, then, using (5), that

$$
\int_\Omega v\, dx = \lim_{t\to 0} \int_\Omega v_t\, dx = 1,
$$

and eventually according to (4), that

$$
|Dv|(\mathbb{R}^n) \leq \liminf_{t\to 0} |Dv_t|(\mathbb{R}^n) \leq h.
$$

Letting $v = v_{|\Omega}$, it follows that $\int_\Omega v\, dx = 1$, and

$$
h \leq |Dv|(\mathbb{R}^n) \leq \liminf_{t\to 0} |Dv_t|(\mathbb{R}^n) = h.
$$

It follows that

$$
\lim_{t\to 0} h_t = h,
$$

and that v is an extremal for h.

We assume from now on that Ω admits a unique Cheeger set $C \subset \Omega$. As a consequence, the only nonnegative normalized extremal for h is $|C|^{-1}\chi_C$; this follows from the fact that every level set of an extremal is a Cheeger set (see [3, Theorem 2]). In particular $u = v = |C|^{-1}\chi_C$. Therefore $v_t \to u$ in $L^1(\Omega)$ and

$$
\lim_{t\to 0} |Dv_t|(\mathbb{R}^n) = |Du|(\mathbb{R}^n).
$$

By [2, Proposition 3.13], this implies that

$$
\lim_{t\to 0} \int_{\mathbb{R}^n} \phi\, d|Dv_t| = \int_{\mathbb{R}^n} \phi\, d|Du|
$$

for any $\phi \in C_c(\mathbb{R}^n)$.

Let us prove the differentiability. Using $w_t = u \circ G_t$ as a test-function for h_t, we obtain

$$
h_t - h \leq \frac{\displaystyle\int_{\mathbb{R}^n} |(DG_t)^T \sigma| \cdot |\det DF_t|\, d|Du|}{\displaystyle\int_\Omega u(y) |\det DF_t(y)|\, dy} - h.
$$

Observe that

$$
|\det DF_t(y)| = 1 + t.\mathrm{div}\, V(y) + o(t),
$$

and

$$
|(DG_t(y))^T \sigma(y)| = |\sigma(y)| - t\langle \sigma(y), DV(y).\sigma(y)\rangle + o(t),
$$

where $o(t)$ is uniform in y. Therefore

$$h_t - h \leq \frac{h + t\int_{\mathbb{R}^n} (\operatorname{div} V - \langle \sigma, DV\sigma\rangle)\, d|Du| + o(t)}{1 + t\int_\Omega u \operatorname{div} V + o(t)} - h$$
$$= \frac{t\left(\int_{\mathbb{R}^n} (\operatorname{div} V - \langle \sigma, DV\sigma\rangle)\, d|Du| - h\int_\Omega u \operatorname{div} V\right)}{1 + t\int_\Omega u \operatorname{div} V + o(t)}.$$

We used the fact that $|\sigma| = 1$ $|Du|$-a.e. and u is a normalized extremal for h. It follows that

$$\limsup_{t\to 0^+} \frac{h_t - h}{t} \leq \int_{\mathbb{R}^n} (\operatorname{div} V - \langle \sigma, DV\sigma\rangle)\, d|Du| - h\int_\Omega u \operatorname{div} V,$$

and

$$\liminf_{t\to 0^-} \frac{h_t - h}{t} \geq \int_{\mathbb{R}^n} (\operatorname{div} V - \langle \sigma, DV\sigma\rangle)\, d|Du| - h\int_\Omega u \operatorname{div} V.$$

Let us now prove the opposite inequality. We use v_t as a test-function for h, and we obtain

$$h_t - h = \int_{\mathbb{R}^n} d|Du_t| - h \geq \int_{\mathbb{R}^n} |(DG_t)^T \sigma_t| \cdot |\det DF_t|\, d|Dv_t| - \frac{\int_{\mathbb{R}^n} d|Dv_t|}{\int_\Omega v_t},$$

where σ_t is such that $Du_t = \sigma_t |Du_t|$. We can also write

$$h_t - h \geq \int_{\mathbb{R}^n} d|Dv_t| + t\int_{\mathbb{R}^n} (\operatorname{div} V - \langle \sigma_t, DV\sigma_t\rangle)\, d|Dv_t| - \frac{\int_{\mathbb{R}^n} d|Dv_t|}{\int_\Omega v_t} + o(t).$$

Since $\operatorname{div} V \in C_c(\mathbb{R}^n)$, we have

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \operatorname{div} V\, d|Dv_t| = \int_{\mathbb{R}^n} \operatorname{div} V\, d|Du|.$$

Observe also that

$$\int_\Omega v_t = 1 - t\int_{\mathbb{R}^n} u_t \operatorname{div} V + o(t) = 1 - t\int_{\mathbb{R}^n} u \operatorname{div} V + o(t).$$

so that

$$\frac{\int_{\mathbb{R}^n} d|Dv_t|}{\int_\Omega v_t} = \int_{\mathbb{R}^n} d|Dv_t| + t\left(\int_{\mathbb{R}^n} d|Dv_t|\right)\left(\int_\Omega u \operatorname{div} V\right) + o(t)$$
$$= \int_{\mathbb{R}^n} d|Dv_t| + th\int_\Omega u \operatorname{div} V + o(t),$$

where we used the fact that $|Dv_t|(\mathbb{R}^n) = h + o(1)$. Hence,

$$h_t - h \geq t\left(\int_{\mathbb{R}^n} \operatorname{div} V\, d|Du| - h\int_\Omega u \operatorname{div} V - \int_{\mathbb{R}^n} \langle \sigma_t, DV\sigma_t\rangle\, d|Dv_t|\right) + o(t)$$

Since $Dv_t \rightharpoonup^* Du$ and $|Dv_t|(\mathbb{R}^n) \to |Du|(\mathbb{R}^n)$, we have, according to Reshetnyak's Theorem (see [2, Theorem 2.39]), that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} f(x, \sigma_t(x))\, d|Dv_t| = \int_{\mathbb{R}^n} f(x, \sigma(x))\, d|Du| \quad \text{ for any } f \in C_b(\mathbb{R}^n \times S^{n-1}).$$

It follows in particular that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \langle \sigma_t, DV\sigma_t\rangle\, d|Dv_t| = \int_{\mathbb{R}^n} \langle \sigma, DV\sigma\rangle\, d|Du|.$$

We thus obtain

$$\limsup_{t\to 0^+} \frac{h_t - h}{t} \geq \int_{\mathbb{R}^n} (\mathrm{div}\, V - \langle \sigma, DV\sigma\rangle)\, d|Du| - h\int_\Omega u \,\mathrm{div}\, V$$

and

$$\liminf_{t\to 0^-} \frac{h_t - h}{t} \leq \int_{\mathbb{R}^n} (\mathrm{div}\, V - \langle \sigma, DV\sigma\rangle)\, d|Du| - h\int_\Omega u \,\mathrm{div}\, V.$$

Therefore

$$h_1(\Omega, V)' = \lim_{t\to 0^+} \frac{h_t - h}{t}$$

exists, and

$$h_1(\Omega, V)' = \int_{\mathbb{R}^n} (\mathrm{div}\, V - \langle \sigma, DV\sigma\rangle)\, d|Du| - h\int_\Omega u \,\mathrm{div}\, V.$$

Since $u = |C|^{-1}\chi_C$, we have that $|Du| = |C|^{-1}\mathscr{H}^{n-1}_{|\partial^* C}$ as a measure. We can thus rewrite the previous formula as

$$\begin{aligned} h_1(\Omega, V)' &= \frac{1}{|C|}\left(\int_{\partial^* C} (\mathrm{div}\, V - \langle \sigma, DV\sigma\rangle)\, d\mathscr{H}^{n-1} - h\int_C \mathrm{div}\, V\right) \\ &= \frac{1}{|C|}\int_{\partial^* C} (\mathrm{div}\, V - \langle \sigma, DV\sigma\rangle - h\langle V, \nu\rangle)\, d\mathscr{H}^{n-1}, \end{aligned}$$

where ν is the unit exterior normal to $\partial^* C$, and σ is given by $Du = \sigma|Du|$. We observe that $\sigma = -\nu$ $\mathscr{H}^{n-1}$ - a.e. on $\partial^* C$. Recall that

$$\mathrm{div}\, V(x) - (\nu(x), DV(x)\nu(x)) = \mathrm{div}_{\partial C} V(x), \qquad x \in \partial^* C,$$

is the tangential divergence of V on $\partial^* C$ (see e.g. [11, Definition 5.4.6]). We thus obtain that

$$h_1(\Omega, V)' = \frac{1}{|C|}\int_{\partial^* C} (\mathrm{div}_{\partial C} V - h\langle V, \nu\rangle)\, d\mathscr{H}^{n-1} \tag{6}$$

which ends the proof of Theorem 1.1. □

Proof of Corollary 1.2. Suppose that Ω admits a unique Cheeger set C which is $C^{1,1}$. The unit exterior normal vector ν to ∂C is thus defined at every point and is Lipschitz continuous. Its components are thus differentiable at $\mathscr{H}^{n-1}$ almost every point of ∂C; moreover, the quantity $\kappa := \mathrm{div}_{\partial C}\nu$ belongs to $L^\infty(\partial C)$ and it can be seen as the distributional curvature of ∂C. Indeed one can easily adapt [11, Section 5.4.3] to the case of $C^{1,1}$ domains to obtain

$$\mathrm{div}_{\partial C} V = \mathrm{div}_{\partial C} V_{\partial C} + \kappa(V, \nu) \qquad \mathscr{H}^{n-1} - a.e.,$$

where $V_{\partial C} = V - (V, \nu)\nu$ is the tangential part of V, and

$$\int_{\partial C} \mathrm{div}_{\partial C} V_{\partial C}\, d\mathscr{H}^{n-1} = 0.$$

Therefore it holds

$$\int_{\partial C} \mathrm{div}_{\partial C} V = \int_{\partial C} \kappa\langle V, \nu\rangle$$

and we can rewrite (6) as

$$\begin{aligned} h_1(\Omega,V)' &= \frac{1}{|C|}\int_{\partial C}(\mathrm{div}_{\partial C} V - h_1(\Omega)\langle V,\nu\rangle)\, d\mathscr{H}^{n-1} \\ &= \frac{1}{|C|}\int_{\partial C}(\kappa - h_1(\Omega))\langle V,\nu\rangle\, d\mathscr{H}^{n-1} \\ &= \frac{1}{|C|}\int_{\partial C\cap\partial\Omega}(\kappa - h_1(\Omega))\langle V,\nu\rangle\, d\mathscr{H}^{n-1} \end{aligned}$$

since $\kappa = h_1(\Omega)$ in $\partial C \cap \Omega$. We then deduce (3). □

We complete this section providing some explicit examples of computation of shape derivatives.

Example 3.1 (The ball)**.** Let $\Omega = B_R$ be the ball of radius R, and V is a vector field such that $V(x) = \nu(x)$ on ∂B_R, we have that $\frac{dh_t}{dt}(0) = \left[\frac{d}{dr}h_1(B_r)\right](R)$. Since $h_1(B_r) = \frac{n}{r}$, we obtain using (3) that

$$h_1(\Omega,V)' = \frac{n\omega_n R^{n-1}}{\omega_n R^n}\cdot\left(\frac{n-1}{R} - \frac{n}{R}\right) = -\frac{n}{R^2}$$

as expected. Now let V be a volume-preserving perturbation; formula (3) becomes

$$h_1(\Omega,V)' = -\frac{1}{|\Omega|}\int_{\partial\Omega}\langle V,\nu\rangle\, d\mathscr{H}^{n-1} = -\frac{1}{|\Omega|}\int_\Omega \mathrm{div}\, V = 0$$

in accordance with the well-known fact that the ball minimizes $h_1(\Omega)$ among all bounded domains with fixed volume.

Example 3.2 (The annulus)**.** As another simple example take $\Omega = A_{r,R} = B_R\backslash\bar{B}_r$, the annulus $\{r < |x| < R\}$, $r < R$. According to [6] and [13], $A_{r,R}$ coincides with its Cheeger set so that

$$h_1(A_{r,R}) = \frac{|\partial A_{r,R}|}{|A_{r,R}|} = n\frac{R^{n-1}+r^{n-1}}{R^n - r^n}.$$

Taking $V(x) = \nu(x)$, we have by direct computation that

$$\begin{aligned} &\frac{d}{dt}h_1(A_{r-t,R+t})_{|t=0} \\ &= n\frac{-R^{2n-2} - r^{2n-2} - (n-1)r^{n-2}R^n - (n-1)R^{n-2}r^n - 2n(rR)^{n-1}}{(R^n - r^n)^2}, \end{aligned}$$

which coincides with formula (3):

$$h_1(\Omega,V)' = \left(\frac{n-1}{R} - h_1(A_{r,R})\right)\frac{|\partial B_R|}{|A_{r,R}|} - \left(\frac{n-1}{r} + h_1(A_{r,R})\right)\frac{|\partial B_r|}{|A_{r,R}|}.$$

In dimension 2 this example can be generalized to curved annulus:

Example 3.3 (Curved annulus in the plane)**.** Let Γ be a smooth planar closed curve with no self-intersection, and $\Omega = \Sigma_{\Gamma,a} = \{x \in \mathbb{R}^2, dist(x,\Gamma) < a\}$ its tubular neighborhood of width a. We take a so small that Ω has no self-intersection. According to [15], $h_1(\Omega) = \frac{1}{a}$ and Ω itself is the unique Cheeger set. We take $V = \nu$. Then $\Omega_t = \Sigma_{\Gamma,a+t}$ and $h(\Omega,V)' = -\frac{1}{a^2} = -h_1(\Omega)^2$ which coincides with formula (3):

$$h_1(\Omega,V)' = \frac{1}{|\Omega|}\int_{\partial\Omega}(\kappa - h_1(\Omega))\, d\mathscr{H}^{n-1}$$

since $\int_{\partial\Omega}\kappa = 2\pi\chi(\Omega) = 0$ according to the Gauss-Bonnet formula.

Example 3.4 (The square)**.** We eventually provide an example where the Cheeger set is a proper subset of Ω. According to [13] a rectangle $R_{a,b} \subset \mathbb{R}^2$ of edges $2a$ and $2b$ has a unique Cheeger set C with

$$h_1(R_{a,b}) = \frac{4-\pi}{2(a+b) - 2\sqrt{(a-b)^2 + \pi ab}} \tag{7}$$

(see e.g. one of the two squares in figure 4). We take $\Omega = [0,1] \times [0,1] = R_{1/2,1/2}$ and $V(x,y) = (\eta(x), 0)$ with $\eta : \mathbb{R} \to [0,1]$ smooth with compact support in $(1-\delta, 1+\delta)$, δ small, and $\eta(x) = 1$ for $x \in (1-\delta/2, 1+\delta/2)$. Then $\Omega_t = (0, 1+t) \times (0,1)$ for sufficiently small t. It follows by direct computations from (7) that

$$h_1(\Omega, V)' = -\frac{1}{2} h_1(\Omega).$$

Since $\partial C \cap \Omega$ is made of arc of circle of radius $1/h_1(\Omega)$, it is easily seen that

$$|C| = 1 - \frac{4-\pi}{h_1(\Omega)^2} = \frac{4\sqrt{\pi} - 2\pi}{4-\pi},$$

$$\mathscr{H}^1(\partial C \cap S) = 1 - \frac{2}{h_1(\Omega)} = \frac{2\sqrt{\pi} - \pi}{4-\pi},$$

where $S := \{1\} \times [0,1]$. It follows that

$$h_1(\Omega, V)' = -h_1(\Omega) \frac{\mathscr{H}^1(\partial C \cap S)}{|C|},$$

which is formula (3) since $\kappa = 0$ on $\partial C \cap \partial \Omega$, $\langle V, \nu \rangle = 1$ on S and $\langle V, \nu \rangle = 0$ on $\partial\Omega \setminus S$.

4. A COUNTER-EXAMPLE TO THE DIFFERENTIABILITY OF $h_1(\Omega)$

If Ω does not admit a unique Cheeger set, then $h_1(\Omega)$ is in general not differentiable. As a counterexample, we consider the "barbell domain", made of two equal rectangles R_1 and R_2 linked by a thin strip (see Figure 4), defined as

$$\Omega = ([0,1] \times [0,1]) \cup ([1,2] \times [0,\varepsilon]) \cup ([2,3] \times [0,1]),$$

where $\varepsilon > 0$ is sufficiently small. Let V be a smooth vector field such that:

- V is supported in $[3-\delta, 3+\delta] \times [-\delta, 1+\delta]$ for some small δ;
- $V(x,y) = f(x,y)\overrightarrow{e_1}$ for some smooth nonnegative function f satisfying $f(3,y) = 1$ for $y \in [0,1]$.

In other words, V shifts the far right edge of Ω to the right. For small positive values of t, $h_1(\Omega_t)$ behaves like the Cheeger constant of a rectangle obtained by enlarging R_2. Recalling formula (7) which gives the Cheeger constant of a rectangle $R_{a,b}$ of edges $2a$ and $2b$, we see that $\frac{\partial}{\partial b} h_1(R_{a,b}) < 0$. Therefore

$$\lim_{t \to 0^+} \frac{h_1(\Omega_t) - h_1(\Omega)}{t} < 0.$$

For small negative values of t, $h_1(\Omega_t) = h_1(R_1) = h_1(\Omega)$ so that

$$\lim_{t \to 0^-} \frac{h_1(\Omega_t) - h_1(\Omega)}{t} = 0.$$

It follows that $h_1(\Omega)$ is not differentiable at $t = 0$.

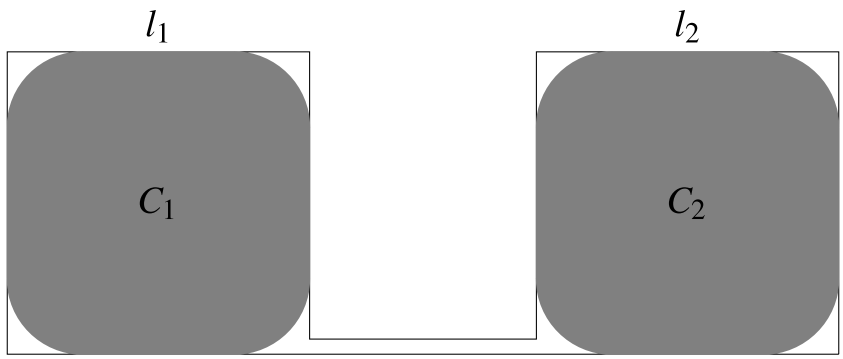

FIGURE 1. If $l_1 = l_2$, the Cheeger sets are given by C_1, C_2 and $C_1 \cup C_2$.

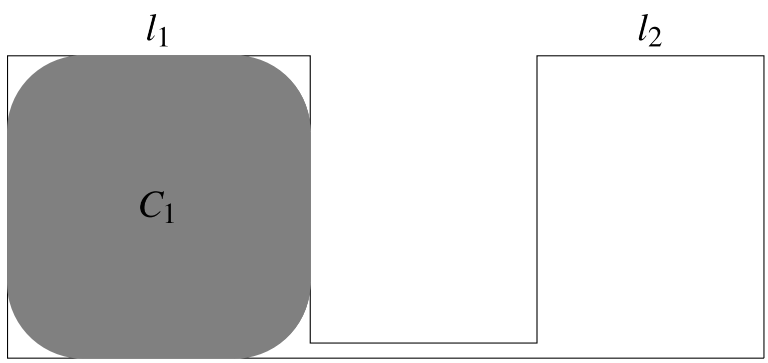

FIGURE 2. If $l_1 > l_2$, the only Cheeger set is given by C_1.

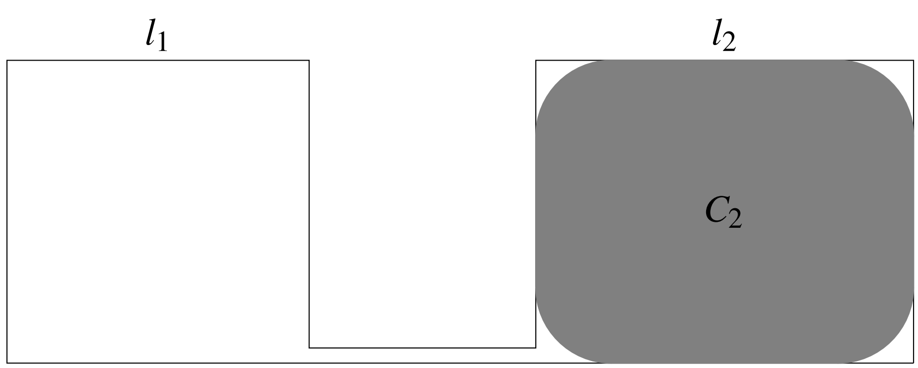

FIGURE 3. If $l_2 > l_1$, the only Cheeger set is given by C_2.

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